Fachhochschule Aargau
Nordwestschweiz

Mathematik Lösungen 3.10

1.) An einer Masse m greift die äussere Kraft F an, welche diesen in Bewegung setzt. Der Bewegung wirkt die geschwindigkeitsproportionale Reibungskraft $F_R = -\mu \cdot v(t)$ entgegen (μ : Reibungskoeffizient). Nach dem Newton'schen Gesetz gilt dann :

$$m \cdot \frac{dv}{dt} = F - \mu \cdot v$$

Berechnen Sie $v(t)$ und die Endgeschwindigkeit $v_{\text{end}} = \lim\limits_{t \to \infty} v(t)$.

$$m \dot{v} = F - \mu \cdot v$$

$$\dot{v} + \frac{\mu}{m} v = \frac{F}{m}$$ lin. DG 1. Ordnung

(h) $F(t) = \int \frac{\mu}{m}\, dt = \frac{\mu}{m} t$

$$v_h(t) = C \cdot e^{-\frac{\mu}{m} \cdot t}$$

(p) $C(t) = \int \frac{F}{m} \cdot e^{\frac{\mu}{m} t}\, dt = \frac{F}{\cancel{m}} \frac{\cancel{m}}{\mu} e^{\frac{\mu}{m} t}$

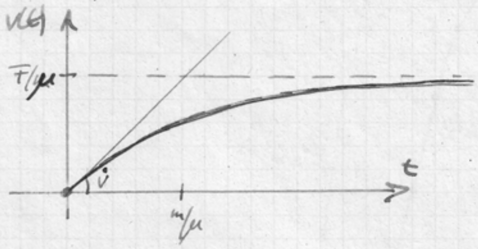
$$\Rightarrow v_p(t) = \frac{F}{\mu}$$

$$\Rightarrow v(t) = C \cdot e^{-\frac{\mu}{m} t} + \frac{F}{\mu}$$

Anfangsbedingung : z.Bsp. $v(t=0) = 0$

$$\Rightarrow C = -\frac{F}{\mu}$$

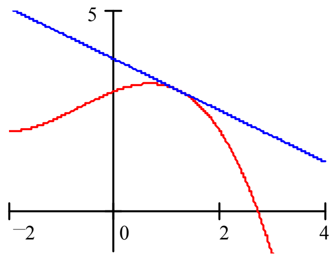
$$\Rightarrow v(t) = \frac{F}{\mu}\left(1 - e^{-\frac{\mu}{m} t}\right)$$



$$\lim_{t \to \infty} v(t) = \frac{F}{\mu}$$

$\dot{v}(t) = \frac{F}{m} \cdot \frac{\mu}{m} e^{-\frac{\mu}{m} t}$ $\quad \dot{v}(0) = \frac{F\mu}{m^2}$

2.) Bei welchen Kurven in der xy–Ebene hat das Trapez, begrenzt durch die x–und y–Achsen, die Tangente und der Ordinate des Tangentenpunktes den Flächeninhalt 1 ?



Fläche: $\dfrac{x\cdot(y+(y-xy'))}{2} = \pm 1$

Test der 4 Fälle !!

$\Rightarrow \quad yx - \dfrac{x^2}{2}y' = \pm 1$

$\Rightarrow \quad y' - \dfrac{2}{x}y = \pm \dfrac{2}{x^2} \qquad$ für $x \neq 0$

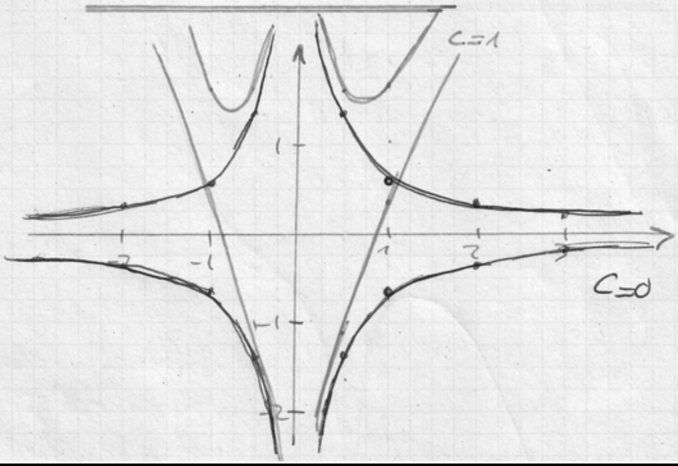
(h) $F(x) = \int -\dfrac{2}{x}\,dx = -2\ln|x|$

$y_h = C\cdot e^{2\ln|x|} = C\cdot x^2$

(p) $C(x) = \int \pm\dfrac{2}{x^2}\,e^{-2\ln|x|}\,dx = \int \pm\dfrac{2}{x^4}\,dx = \mp\dfrac{2}{3\cdot x^3}$

$y_p = \mp\dfrac{2}{3\cdot x^3}\,x^2 = \mp\dfrac{2}{3\cdot x}$

$\Rightarrow \quad y(x) = Cx^2 \pm \dfrac{2}{3x}$



3.) Lösen Sie die folgenden Anfangswertprobleme :

a.) $4y'' + 12y' + 9y = 1$ mit $y(0) = 1$, $y'(0) = 1$

$4y'' + 12y' + 9y = 1$ mit $y(0) = 1 = y'(0)$

$0 = 4\lambda^2 + 12\lambda + 9 = (2\lambda + 3)^2 \Rightarrow \lambda_{1,2} = -\frac{3}{2}$

$y_h(x) = (C_1 + C_2 x) e^{-\frac{3x}{2}}$

$y_p(x) = C_3 \qquad y_p' = 0 \Rightarrow 9C_3 = 1$

$\Rightarrow \underline{y(x) = (C_1 + C_2 x) e^{-\frac{3}{2}x} + \frac{1}{9}}$

$y(0) = C_1 + \frac{1}{9} = 1 \Rightarrow C_1 = \frac{8}{9}$

$y'(0) = -\frac{3}{2} C_1 + C_2 = 1 \Rightarrow C_2 = 1 + \frac{3}{2} \cdot \frac{8}{9} = \frac{7}{3}$

$\underline{y(x) = \left(\frac{8}{9} + \frac{7}{3}x\right) e^{-\frac{3x}{2}} + \frac{1}{9}}$

b.) $6y'' + 7y' + 2y = 6$ mit $y(0) = 0$, $y'(0) = 1$

$6y'' + 7y' + 2y = 6$ mit $y(0) = 0$; $y'(0) = 1$

$0 = 6\lambda^2 + 7\lambda + 2 \Rightarrow \qquad \lambda_{1,2} = \frac{-7 \pm \sqrt{49 - 48}}{12} = \frac{-7 \pm 1}{12}$

$\lambda_1 = -\frac{1}{2} \qquad \lambda_2 = -\frac{2}{3}$

$y_h(x) = C_1 \cdot e^{-\frac{x}{2}} + C_2 \cdot e^{-\frac{2}{3}x}$

$y_p = C_3 \Rightarrow 2C_3 = 6 \qquad C_3 = 3$

$\underline{y(x) = C_1 \cdot e^{-\frac{x}{2}} + C_2 \cdot e^{-\frac{2}{3}x} + 3}$ allg. Lösung

$y(0) = C_1 + C_2 = -3$

$y'(0) = -\frac{C_1}{2} - \frac{2}{3} C_2 = 1 \qquad 2$

$\left(1 - \frac{4}{3}\right) C_2 = -1 \Rightarrow C_2 = \frac{-1 \cdot 3}{-1} = 3$

$\Rightarrow C_1 = -6$

$\Rightarrow \underline{y(x) = -6e^{-\frac{x}{2}} + 3e^{-\frac{2}{3}x} + 3}$

c.) $y'' - y = e^{-2x}$ mit $y(0) = 0$, $y'(0) = 0$

$$y'' - y = e^{-2x} \quad \text{mit } y(0) = 0 = y'(0)$$

$$\lambda^2 - 1 = 0 \Rightarrow \lambda_{1/2} = \pm 1$$

$$y_h(x) = C_1 e^{x} + C_2 e^{-x}$$

$$y_p = C_3 \cdot e^{-2x} \qquad y'_p = -2C_3 e^{-2x}$$
$$y''_p = +4C_3 e^{-2x}$$

$$\Rightarrow 4C_3 e^{-2x} - C_3 e^{-2x} = e^{-2x} \Rightarrow C_3 = \frac{1}{3}$$

$$\Rightarrow \underline{y(x) = C_1 \cdot e^{x} + C_2 e^{-x} + \frac{1}{3} e^{-2x}}$$

$$\left.\begin{aligned} y'(0) = C_1 - C_2 - \frac{2}{3} &= 0 \\ C_1 + C_2 + \frac{1}{3} &= 0 \end{aligned}\right\} \quad 2C_1 = \frac{1}{3}$$
$$C_2 = C_1 - \frac{2}{3} = \frac{1}{6} - \frac{4}{6} = -\frac{1}{2}$$

4.) Lösen Sie die folgenden Anfangswertprobleme :

a.) $\ddot{y} + 4y = \cos(2t)$ mit $y(0) = -1$, $\dot{y}(0) = 1/2$

$$\ddot{y} + 4y = \cos 2t \quad \text{mit } y(0) = -1, \ \dot{y}(0) = \frac{1}{2}$$

$$\lambda^2 + 4 = 0 \Rightarrow \lambda_{1/2} = \pm i2$$

$$\Rightarrow y_h(t) = (C_1 \cos 2t + C_2 \sin 2t)$$

Resonanzfall !

$$y_p(t) = (C_3 \cos 2t + C_4 \sin 2t)\, t$$

$$\dot{y}_p(t) = C_3 \cos 2t - 2C_3 t \sin 2t + C_4 \sin 2t + 2C_4 t \cos 2t$$

$$\ddot{y}_p(t) = -2C_3 \sin 2t - 2C_3 \sin 2t - 4C_3 t \cos 2t + 2C_4 \cos 2t + 2C_4 \cos 2t - 4C_4 t \sin 2t$$

$$\ddot{y}_p + 4y = -4C_3 \sin 2t \cancel{- 4C_3 t \cos 2t} + 4C_4 \cos 2t \cancel{- 4C_4 t \sin 2t} \cancel{+ 4C_3 t \cos 2t} \cancel{+ 4C_4 t \sin 2t}$$
$$= \cos 2t$$

$$\Rightarrow C_3 = 0 \qquad C_4 = \frac{1}{4}$$

$$\Rightarrow \underline{\underline{y(t) = C_1 \cos 2t + C_2 \sin 2t + \frac{t}{4} \sin 2t}}$$

$$y(0) = C_1 = -1$$

$$\dot{y}(0) = -2C_1 \sin 2t + 2C_2 \cos 2t + \frac{\sin 2t}{4} + \frac{2t}{4} \cos 2t \Big|_{t=0}$$
$$= 2C_2 = \frac{1}{2} \Rightarrow C_2 = \frac{1}{4}$$

$$\Rightarrow \underline{\underline{y(t) = -\cos 2t + \frac{1}{4} \cdot \sin 2t\,(1+t)}}$$

b.) $\ddot{x} + 2\dot{x} + x = -2 \cdot \sin(t)$ mit $x(0) = x_0$, $\dot{x}(0) = -1$

$$\ddot{x} + 2\dot{x} + x = -2 \cdot \sin t \qquad \text{mit } x(0) = x_0, \quad \dot{x}(0) = -1$$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda + 1)^2 = 0 \quad \Longleftrightarrow \quad \lambda_{1/2} = -1$$

$$x_h(t) = (C_1 + C_2 t) \cdot e^{-x}$$

$$g(t) = -2 \cdot \sin t = e^{0 \cdot t} \cdot (-2 \cdot \sin t) = e^{0 \cdot t}(2 \cdot \sin(-t))$$

$$\Rightarrow x_p(t) = \tilde{C}_1 \sin t + \tilde{C}_2 \cos t$$

$$\dot{x}_p = \tilde{C}_1 \cos t - \tilde{C}_2 \sin t$$

$$\ddot{x}_p = -\tilde{C}_1 \sin t - \tilde{C}_2 \cos t$$

$$\begin{aligned} \ddot{x}_p + 2\dot{x}_p + x = & -\cancel{\tilde{C}_1 \sin t} - \cancel{\tilde{C}_2 \cos t} \\ & -2\tilde{C}_2 \sin t + 2\tilde{C}_1 \cos t \\ & +\cancel{\tilde{C}_1 \sin t} + \cancel{\tilde{C}_2 \cos t} = -2 \sin t \end{aligned}$$

$$\Rightarrow \tilde{C}_1 = 0, \quad \tilde{C}_2 = 1$$

$$\Rightarrow \underline{x(t) = (C_1 + C_2 t)\, e^{-t} + \cos t} \qquad \text{allg. Lösung}$$

$$x(0) = C_1 + 1 = x_0 \quad \Rightarrow \quad C_1 = x_0 - 1$$

$$\dot{x}(t) = -C_1 \cdot e^{-t} + C_2 e^{-t} - C_2 t \cdot e^{-t} - \sin t$$

$$\dot{x}(0) = -C_1 + C_2 = -1$$

$$C_2 = -1 + C_1 = -1 + x_0 - 1 = x_0 - 2$$

$$\underline{\underline{x(t) = \left[(x_0 - 1) + (x_0 - 2)t\right] e^{-t} + \cos t}}$$

5.) Lösen Sie aus dem Buch „Numerische Methoden“ von Faires & Burden aus dem Abschnitt 3.4 die Aufgabe 7.).

Vgl. Maple–Vorlage